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The collapse of a viscous tube

By J. A. LEWIS

Bell Laboratories, Murray Hill, New Jersey 07974

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The collapse of a composite circular viscous tube owing to external pressure and surface tension is considered. It is shown that, for small surface tension, the collapse time, at which the tube closes, is very sensitive to the viscosity of the inner tube.

1. Introduction

We consider the radial collapse of a composite viscous tube owing to external pressure and surface tension. This problem arises in the manufacture of optical fibre, which consists of a 'core' of high refractive index surrounded by a silica 'cladding'. The fibre is drawn from a composite cylindrical billet, or 'preform', softened in a furnace or laser beam. This preform is fabricated by depositing the high index glass on the inside of a silica tube which is then collapsed by successive passes of a ring of gas flames (four or more burners being used) while being rotated in a glass lathe to preserve symmetry. It is this collapse process, shown schematically in figure 1, which we study here.

In order to obtain results of use to the process designer, we consider only the simple problem of the collapse of a composite tube with piecewise constant viscosity. Properly we should simultaneously calculate the fluid motion and the temperature distribution, which are coupled by the strong dependence of the viscosity on temperature typical of glassy materials. This problem is further complicated by radiative transfer in the partially transparent, white-hot material, silica softening at a temperature of the order of 2000 °K. At this stage it seems expedient to bypass this vastly more complicated problem.

The collapse of the composite tube, with piecewise constant viscosity, is described, as we shall see, by a first-order ordinary differential equation which is soluble, in general numerically, by quadratures. In spite of this simplicity, the process presents some novel features, which we now discuss.

(i) *Collapse mechanism.* One might suppose that collapse is caused by surface tension, resisted by the viscosity of the tube as it is softened in the gas flame. A simple order-of-magnitude argument, however, suffices to show that this is not the case. From the surface tension T, the overpressure Δp, a typical viscosity μ and some typical dimension, e.g. the initial inner radius a_0 of the tube, we can form two typical times: a 'surface-tension collapse time' $\mu a_0/T$ and a 'viscous collapse time' $\mu/\Delta p$. If we insert typical values, namely $a_0 \sim 0{\cdot}5$ cm, $T \sim 200$ dyne/cm and $\mu \sim 10^6$ P (Bacon, Hasapis & Wholley 1959), we find a surface-tension collapse time of the order of 2500 s, a sizeable fraction of an hour, whereas the actual process takes place in a matter of minutes. Now in fact the ring of gas flames exerts a pressure on the rotating tube

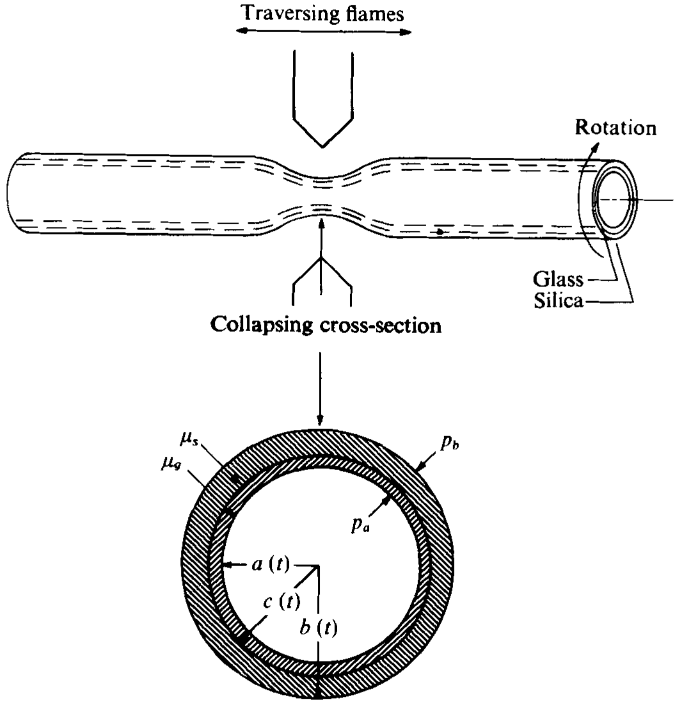

FIGURE 1. The collapse process.

whose magnitude may be estimated from the experimental observation that an increase in internal pressure of about 1 in. of water ($\sim 2{\cdot}5 \times 10^3$ dyne/cm^2) is sufficient to stop collapse. This gives a collapse time of about 400 s, more in line with experience.

(ii) *Core viscosity*. This study was initiated by the experimental observation that the collapse time of a composite silica–glass tube is considerably less than that of a homogeneous silica tube of the same dimensions. A simple explanation is not hard to find. Consider first the purely viscous collapse of a homogeneous tube of viscosity μ_s owing to the overpressure Δp alone. The collapse time is ultimately determined by the balance between viscous stress, of order $\mu_s \dot{a}/a$ at the inner radius a, and the overpressure Δp. Purely viscous collapse is then exponential, with $a \rightarrow 0$ only as $t \rightarrow \infty$. In a composite tube the same result applies except that μ_s is replaced by μ_g, the core viscosity. However, when μ_g is zero, the core material serves merely as a pressure transmitter, offering no resistance to the motion, which must cease when the incompressible core material fills the original void. Thus, for purely viscous collapse, the collapse time is infinite for any positive core viscosity μ_g, no matter how small, but finite when $\mu_g = 0$. The presence of a small surface tension removes this infinity, replacing it, as we shall see, with a finite collapse time of order $(\mu_g/\Delta p)\ln(a_0\,\Delta p/T)$, which exhibits a similar sensitivity to the value of μ_g for small T.

Figure 2 illustrates this state of affairs for a typical preform which has a diameter $2b_c = 1$ cm, a core area one-half the total area ($\lambda^2 = (c_c/b_c)^2 = 0{\cdot}5$) and which is formed from a composite tube with an initial inside diameter equal to the preform diameter ($\alpha_0 = a_0/b_c = 1$). Since drawing preserves geometrical similarity, λ is the same for both the fibre and the preform. For a given fibre λ is then fixed and b_c gives the length

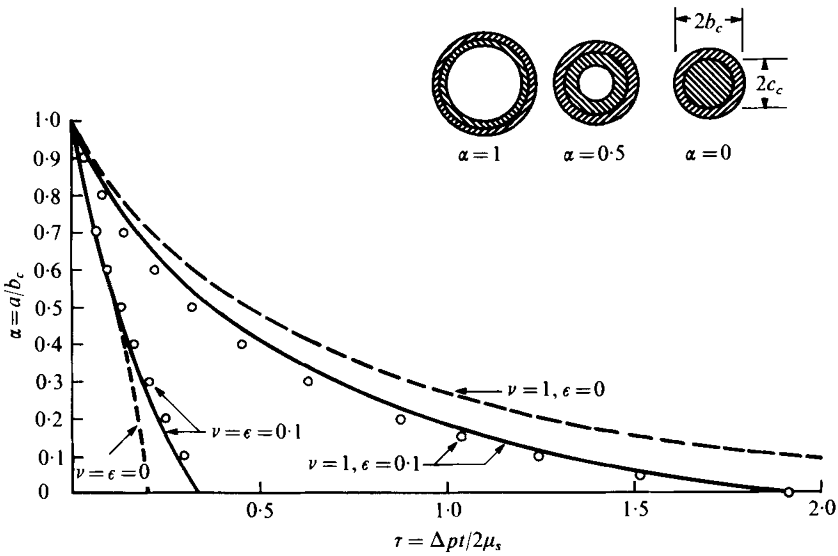

FIGURE 2. The inner radius of a composite viscous tube as a function of time.

of fibre (proportional to b_c^2) which may be drawn from a preform of given length. It is thus convenient to use b_c as the length scale, as in figure 2.

The collapsing composite tube, which initially has $\alpha = \alpha_0 = 1$, passes through all possible initial configurations with thicker walls, for example the composite tube with $\alpha = \frac{1}{2}$, sketched in the figure. Larger values of α_0 correspond to tubes with very thin walls with large initial collapse rates. For example, if α_0 is increased to 5 the dimensionless collapse time of the homogeneous tube considered in figure 2 ($\nu = \mu_g/\mu_s = 1$, $\epsilon = T/b_c \Delta p = 0{\cdot}1$) is increased from $\tau_c = 2{\cdot}00$ to only $\tau_c = 2{\cdot}29$.

The solid curves in figure 2 are exact solutions of the differential equation for collapse of a homogeneous tube ($\nu = 1, \epsilon = 0{\cdot}1$) and of a composite tube ($\nu = \epsilon = 0{\cdot}1$). The upper dashed curve ($\nu = 1$, $\epsilon = 0$) shows purely viscous collapse of the homogeneous tube. In this case $\alpha \to 0$ only as $\tau \to \infty$. On the other hand, the lower dashed curve ($\nu = \epsilon = 0$) shows the limiting case of a resistanceless core without surface tension. Even with $\nu = 0{\cdot}1$, a very modest decrease in viscosity for typical glasses, one begins to approach the limiting curve.

In figure 2 we have assumed a surface tension $T = 200$ dyne/cm and an overpressure $\Delta p = 4 \times 10^3$ dyne/cm^2, to give the value $\epsilon = 0{\cdot}1$ for $b_c = 0{\cdot}5$ cm. Provided that ϵ is held fixed by varying Δp, the curves in figure 2 apply to any other preform diameter, but of course with a different typical time $2\mu_s/\Delta p$. The circles indicate the approximate solution

$$\tau \approx \bar{\tau} = \frac{\nu}{1+\epsilon} \ln \frac{(1+\epsilon)\alpha_0 + \epsilon}{(1+\epsilon)\alpha + \epsilon} + (1-\nu) \ln \frac{\gamma_0}{\gamma} - \ln \frac{\beta_0}{\beta}, \tag{1}$$

valid for small ϵ, where $\alpha = a/b_c$, $\gamma = c/b_c$ and $\beta = b/b_c$ for an inner radius a, intermediate radius c and outer radius b. The corresponding collapse time, at which $\alpha = 0$, is then approximated by

$$\tau_c \approx \bar{\tau}_c = \frac{\nu}{1+\epsilon} \ln \frac{(1+\epsilon)\alpha_0 + \epsilon}{\epsilon} + (1-\nu) \ln \frac{\gamma_0}{\lambda} - \ln \beta_0, \tag{2}$$

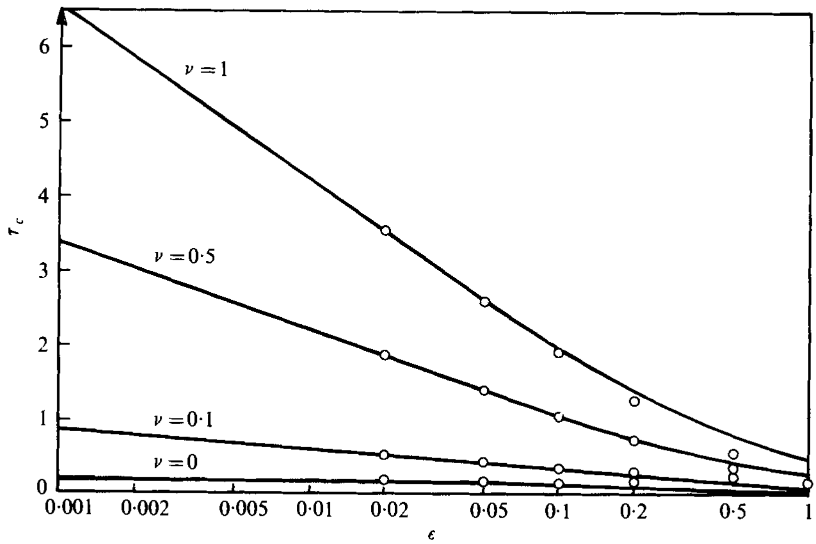

FIGURE 3. The collapse time as a function of the small surface-tension parameter ϵ.

so that τ_c tends to infinity like $\ln(1/\epsilon)$ as ϵ tends to zero. Figure 3 shows τ_c and $\bar{\tau}_c$ (circles) as functions of ϵ for $\lambda^2 = 0{\cdot}5$ and $\alpha_0 = 1$ for the cases $\nu = 0{\cdot}1$, $0{\cdot}5$ and $1{\cdot}0$ and the limiting case $\nu = 0$.

In the following, we first derive an ordinary differential equation for the collapse motion having the dimensionless form

$$d\tau/d\alpha = -f(\alpha).$$

In terms of $\tau' = \tau_c - \tau$, this can be integrated, in general numerically, backwards from $\alpha = 0$ until the given value of α_0 is reached. The approximate solution is obtained by combining two solutions, one valid for the initial, viscous collapse near $\alpha = \alpha_0$, the other for final closure with small surface tension.

The procedure amounts technically to the determination of a composite expansion from two matched asymptotic expansions in ϵ, one near $\alpha = 0$, in terms of a 'stretched variable' $\tilde{\alpha} = \alpha/\epsilon$, the other for large α. However, as the approximation is easily derived from a simple physical argument and the exact solution is expressed as an explicit integral over α, it seems easier to estimate the approximation error directly than to apply the matching technique. The factors of $1+\epsilon$ in (1) and (2), which make the error of order ϵ, are introduced to make the error calculation simpler. Their inclusion amounts to a first approximation of the effect of finite tube diameter on final closure, the zeroth approximation being simply a hole in an infinite medium. For example, in their absence (2) gives

$$\tau_c \approx \nu \ln \frac{\alpha_0 + \epsilon}{\epsilon} + (1-\nu) \ln \frac{\gamma_0}{\lambda} - \ln \beta_0,$$

a simpler approximation which amounts to extending the straight, left-hand portions of the curves in figure 3 to the right: not a very great change, at least for $\epsilon < 0{\cdot}1$.

In the following we first derive the differential equation describing collapse from the equations governing viscous flow. We then use a simple physical argument to derive (1) and (2). Finally we calculate the error in the approximations so obtained.

2. The collapse equation

In the radial motion of a viscous incompressible fluid the velocity $u(r,t)$ and pressure $p(r,t)$ satisfy the equations

$$\partial(ru)/\partial r = 0, \tag{3}$$

$$\partial\tau_{rr}/\partial r = (\tau_{\theta\theta}-\tau_{rr})/r, \tag{4}$$

where the radial and circumferential stresses τ_{rr} and $\tau_{\theta\theta}$ are given by

$$\tau_{rr} = -p+2\mu\,\partial u/\partial r, \quad \tau_{\theta\theta} = -p+2\mu u/r$$

for a viscosity $\mu = \mu(r,t)$. In (4) we have already neglected the fluid inertia in the present low speed, high viscosity flow. It is an easy matter, and an unnecessary frill, to take it into account by inserting an initial 'acceleration boundary layer' near $t = 0$.

These equations hold in $a(t) < r < b(t)$, while on $r = a$

$$u = \dot{a}, \quad \tau_{rr} = -p_a+T/a,$$

for an internal pressure p_a and surface tension T, and on $r = b$

$$u = \dot{b}, \quad \tau_{rr} = -p_b-T/b,$$

for an external pressure p_b. Finally we have the initial conditions

$$a(0) = a_0, \quad b(0) = b_0$$

and the overall incompressibility condition

$$b^2-a^2 = b_0^2-a_0^2.$$

Equation (3) and the boundary conditions on u imply that

$$u = a\dot{a}/r,$$

so that

$$\partial\tau_{rr}/\partial r = 4\mu a\dot{a}/r^3.$$

Integration yields

$$\tau_{rr} = (\tau_{rr})_a+4a\dot{a}\int_a^r \mu\,dr/r^3$$

and use of the boundary conditions on τ_{rr} yields the collapse equation

$$4a\dot{a}\int_a^b \mu\,dr/r^3+T(1/a+1/b) = -\Delta p, \tag{5}$$

where $\Delta p = p_b-p_a$. We now assume that

$$\mu = \begin{cases}\mu_g & \text{for} \quad a(t) < r < c(t),\\ \mu_s & \text{for} \quad c(t) < r < b(t),\end{cases}$$

where μ_g is the constant viscosity of the glassy 'core', μ_s the constant viscosity of the silica 'cladding', and the core radius c satisfies the incompressibility condition

$$c^2-a^2 = c_0^2-a_0^2.$$

In this case (5) becomes

$$2\mu_s a\dot{a}\left[\frac{1}{c^2}-\frac{1}{b^2}+\nu\left(\frac{1}{a^2}-\frac{1}{c^2}\right)\right]+T\left(\frac{1}{a}+\frac{1}{b}\right) = -\Delta p, \tag{6}$$

where $\nu = \mu_g/\mu_s$ and

$$b^2 = b_0^2-a_0^2+a^2 = b_c^2+a^2,$$

$$c^2 = c_0^2-a_0^2+a^2 = c_c^2+a^2$$

for a final cladding (preform) radius b_c and core radius c_c.

3. Small surface tension

We now seek an approximate solution of (6) for small T. First we note that, when $T = 0$, (6) has the solution (Hint: $a\dot{a} = b\dot{b} = c\dot{c}$)

$$t = (2\mu_s/\Delta p)\,[\nu \ln (a_0/a) + (1-\nu) \ln (c_0/c) - \ln (b_0/b)]. \tag{7}$$

This should give a good approximation for large a, i.e. when $T/a \ll \Delta p$. As we would expect for purely viscous collapse, $a \to 0$ only for $t \to \infty$, except of course in the case of a resistanceless, incompressible core ($\nu = 0$).

On the other hand, near closure, as $a \to 0$, $b \to b_c$ and $c \to c_c$, a satisfies the equation for the collapse of a hole in an infinite homogeneous medium under pressure, namely

$$2\mu_s \nu \dot{a}/a + T/a = -\Delta p, \tag{8}$$

with the solution $\quad a = -T/\Delta p + (a_c + T/\Delta p) \exp(-\Delta p t/2\mu_s \nu)$

or

$$t = (2\mu_s/\Delta p)\, \nu \ln \frac{a_c + T/\Delta p}{a + T/\Delta p}, \tag{9}$$

where t is measured from the time at which a has the small value a_c. This suggests somehow superposing (7) and (9) to obtain a solution valid for all a. Actually we need only observe that, if we replace a_c by a_0 in (9), then, when $T/a_0\Delta p \ll 1$, (9) gives the first term of (7) as $a \to a_0$. Thus a first approximation for all a is

$$t \approx (2\mu_s/\Delta p)\left[\nu \ln \frac{a_0 + T/\Delta p}{a + T/\Delta p} + (1-\nu) \ln \frac{c_0}{c} - \ln \frac{b_0}{b}\right]. \tag{10}$$

Equation (10) gives a useful approximation for most practical purposes. For example, it exhibits the observed sensitivity to core viscosity, for when $T/a_0\Delta p \ll 1$ the collapse time t_c, at which $a = 0$, is of order

$$t_c \sim (2\mu_s/\Delta p)\, \nu \ln (a_0 \Delta p/T),$$

which is directly proportional to $\mu_g = \nu\mu_s$. We can, however, improve (10) slightly with very little additional effort and this we do, not so much to improve the accuracy of the approximate solution, but to facilitate our subsequent error calculations.

Whereas we obtained (8) by neglecting both $1/b^2$ and $1/c^2$ compared with $1/a^2$ and $1/b$ compared with $1/a$, we can obtain a first approximation to the effect of finite tube diameter by replacing $1/b$ by $1/b_c$. This gives the same solution except that Δp is replaced everywhere by $\Delta p + T/b_c$. Equation (10) thus becomes

$$t \approx \bar{t} = (2\mu_s/\Delta p)\left[\frac{\nu}{1+\epsilon} \ln \frac{\alpha_0(1+\epsilon)+\epsilon}{\alpha(1+\epsilon)+\epsilon} + (1-\nu) \ln \frac{\gamma_0}{\gamma} - \ln \frac{\beta_0}{\beta}\right], \tag{11}$$

which is (1) with $\quad \epsilon = T/b_c\Delta p, \quad \alpha = a/b_c, \quad \gamma = c/b_c = (\lambda^2 + \alpha^2)^{\frac{1}{2}},$

$$\beta = b/b_c = (1+\alpha^2)^{\frac{1}{2}}, \quad \lambda = c_c/b_c.$$

4. An error estimate

It remains to show that (11) is a *bona fide* approximation, i.e. that, for all α, $t-\bar{t}$, or rather the dimensionless time difference $\tau-\bar{\tau} = (\Delta p/2\mu_s)\,(t-\bar{t})$, vanishes as $\epsilon \to 0$. This we do by direct computation. First we note that (6) can be put in the dimensionless form

$$d\tau/d\alpha = -f(\alpha),$$

where

$$f(\alpha) = [(1-\lambda^2)\,\alpha^2 + \nu\lambda^2\beta^2]/\beta\gamma^2[\alpha\beta + \epsilon(\alpha+\beta)],$$

so that

$$\tau = \int_{\alpha}^{\alpha_0} f(\alpha)\,d\alpha.$$

Similarly

$$\bar{\tau} = \int_{\alpha}^{\alpha_0} \bar{f}(\alpha)\,d\alpha,$$

where

$$\bar{f}(\alpha) = \frac{\nu}{(1+\epsilon)\,\alpha+\epsilon} + \frac{(1-\nu)\,\alpha}{\gamma^2} - \frac{\alpha}{\beta^2}$$

and we have again made use of the identities $\alpha\,d\alpha = \beta\,d\beta = \gamma\,d\gamma$. For small ϵ,

$$f - \bar{f} = \epsilon\,\frac{\nu(1+\alpha)\,\beta^3\gamma^2 - (\alpha+\beta)\,[(1-\lambda^2)\,\alpha^2 + \nu\lambda^2\beta^2]}{\alpha^2\beta^3\gamma^2},$$

plus terms which vanish more rapidly than ϵ as $\epsilon \to 0$. This is clearly bounded when $\alpha > 0$. The critical case is then $\alpha \to 0$, for which one can show that the numerator is of order α^2, so that $f-\bar{f}$ is bounded as $\alpha \to 0$ and $\tau-\bar{\tau}$ is of order ϵ for small ϵ for all $\alpha \geqslant 0$.

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